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SEEPAGE REFRACTION IN A SEMICIRCULAR LENS LOCATED AT THE BOUNDARY OF TWO POROUS MASSIFS†

Yu. V. OBNOSOV

Kazan'

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The problem of calculating the two-dimensional seepage field in a structurally inhomogeneous three-component medium in the form of two infinitely porous massifs with a semicircular inclusion in their plane boundary of contact is considered. The distribution of the seepage rate, when two matching conditions along the lines of contact of unlike zones are strictly satisfied, is obtained in closed analytic form by methods of complex analysis. Limiting cases of the conduction of the components of the medium and cases of the degeneration of a three-component medium into a two-component medium are considered.

In seepage problems, it is important to find the distribution of the flow rate and on the basis of this, to estimate the local stability along the joining lines of porous media of differing coarseness and of natural or artificial origin (the stratified zones of inhomogeneity, the centres of embankments, screens, reverse filters, etc.) [1, 2]. There are very few strictly accurate solutions of such problems in the case of structures which are piecewise-homogeneous with respect to their permeability. The application of the technique of boundary-value problems in the theory of analytic functions to seepage problems in inhomogeneous media was initiated by Polubarinova-Kochina and her disciples [3], who considered flow patterns under the apron of a dam in a multilayered earth, flows in a reservoir with circular and elliptic inclusions, etc.

Below, we develop the ideas proposed in [3] and [4–8] as applied to a seepage scheme in a three-component medium consisting of two porous massifs and a semicircular lens on their contact boundary. At infinity, the seepage rate vector in one of the two massifs is specified to be $\mathbf{V}_0 = (V_{0x}, V_{0y})$. In practice, such a situation occurs in the body of earth reservoirs where the foundations are laid layer by layer and the lens simulates one of the possible defects in such a stacking [2]. When there is no lens, the flow is linear with refraction along the contact boundary between the two half-planes. Generally speaking, the lens leads to the occurrence of a two-dimensional field in all three media.

1. FORMULATION OF THE PROBLEM

It is well known [3] that the problem of plane steady-state seepage in accordance with Darcy's law in a medium for which the seepage coefficient $k(x, y)$ is a piecewise-constant function ($k(x, y) \equiv k_j =$ const when $(x, y) \in D_j$) is described by Laplace's equation for the pressure head $h(x, y) = h_j(x, y)$, $(x, y) \in D_j$ in the zones of homogeneity D_j

$$\Delta h_j(x, y) = 0$$

and the matching conditions

$$h_j = h_m,\ \partial h_j / \partial s = \partial h_m / \partial s$$

along the contact boundary between the dissimilar zones D_j and D_m. The following problem is obtained by introducing the complex variable of the physical plane of flow $z = x + iy$, the complex potential $w(z) = \varphi + i\psi$ ($\varphi = -kh$ is the potential and ψ is the stream function which is harmonically conjugate to φ) and the seepage rate vector $\mathbf{v} = (\upsilon_x, \upsilon_y) = \upsilon_x + i\upsilon_y = \overline{dw/dz} = -k\nabla h$ in relation to the latter potential

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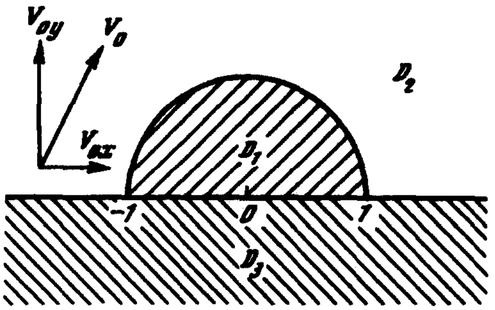

Fig. 1.

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{rot} \mathbf{v} = 0, \quad z \in D_j$$

$$(\mathbf{v}_j)_n = (\mathbf{v}_m)_n, \quad (\mathbf{v}_j)_\tau / k_j = (\mathbf{v}_m)_\tau / k_m, \quad z \in dD_j \cap \partial D_m$$

The first two of these equalities indicate that the function $\upsilon(z) = \upsilon_x - i\upsilon_y$, which is the complex conjugate of the vector $\mathbf{v}(z) = \upsilon_x + i\upsilon_y$, is holomorphic in D_j, and the second pair of equalities expresses the continuity of the normal and the proportionality of the tangential constituent components of the vector $\mathbf{v}$ along the boundary separation of the zones with different conduction coefficients.

It is well known that corresponding problems in the theory of heterogeneous media in electrodynamics [9, 10] ($\mathbf{v}$ is the actual density vector and k is the electrical conductivity, a quantity which is the reciprocal of the resistivity of the medium), in magnetodynamics [11] ($\mathbf{v}$ is the electric field strength vector and k is the magnetic permeance), in problems of the antiplane deformation theory of elasticity [12, 13] (h is the displacement and k is the shear modulus, in diffusion theories [14] (h is the concentration and k is the diffusion coefficient, heat conduction [15] ($\mathbf{v}$ is the thermal flux vector and k is the thermal conductivity), etc. lead to exactly the same mathematical model.

The problem under consideration is equivalent [9] to the problem of $\mathbb{R}$-linear union (this is also referred to as the generalized Riemann problem or the Markushevich problem). In the situation being considered here this problem consists of constructing a function $\upsilon(z) = \upsilon_j(z) = \upsilon_{jx}(x, y) - i\upsilon_{jy}(x, y)$, $z \in D_j$, ($j = 1, 2, 3$) which is holomorphic in each of the homogeneous components D_j (Fig. 1) and continuous in their closure everywhere with the exception of perhaps just the corner points $z = \pm 1$, where the existence of integrable singularities is permitted, using the boundary condition

$$\upsilon_1(t) = A\upsilon_2(t) + Bt^{-2}\overline{\upsilon_2(t)}, \quad t \in l = \{t: |t| = 1, \ \operatorname{Im} t > 0\} \tag{1.1}$$

$$\upsilon_3(x) = A_{1,2}\upsilon_{1,2}(x) - B_{1,2}\overline{\upsilon_{1,2}(x)}, \quad x \in \partial D_3 \cap \partial D_{1,2} \setminus \{-1, 1\}$$

In addition to this condition, it is specified that

$$\upsilon_2(\infty) = V_0 = V_{0x} - iV_{0y} \tag{1.2}$$

The notation

$$A = \frac{k_1 + k_2}{2k_2}, \quad B = 1 - A; \quad A_{1,2} = \frac{k_3 + k_{1,2}}{2k_{1,2}}, \quad B_{1,2} = 1 - A_{1,2} \tag{1.3}$$

is used in (1.1), where k_j is the seepage coefficient of the medium D_j.

Below, we shall carry out a full investigation of problem (1.1)–(1.3) and construct, in explicit analytic form, expressions for the seepage rate vector and the complex potential without constraints of any kind on the values of the conductance of the components of the medium and with an arbitrarily orientated external field vector.

2. THE SOLUTION OF PROBLEM (1.1), (1.2) IN NON-LIMITING AND NON-DEGENERATE CASES

It is assumed everywhere in this section that $k_j \neq 0, \infty$ and $k_j \neq k_m$ when $j \neq m$ ($j, m = 1, 2, 3$), that is, both the cases when the medium contains ideally draining or water-impermeable components and when the three-component medium in question degenerates into a two-component medium are temporarily excluded from the treatment. With the assumptions which have been made, it is obvious that the quantities

$$\Delta = B / A, \quad \Delta_{1,2} = B_{1,2} / A_{1,2} \tag{2.1}$$

satisfy the inequalities: $0 < |\Delta|, |\Delta_{1,2}| < 1$.

Suppose D_j^* is a domain which is symmetric with the domain D_j with respect to the real axis. It is clear that each of the functions

$$F_{1,2}(z) = \begin{cases} A_{1,2}\upsilon_{1,2}(z), & z \in D_{1,2} \\ \upsilon_3(z) + B_{1,2}\overline{\upsilon_{1,2}(\bar{z})}, & z \in D_{1,2}^* \end{cases} \tag{2.2}$$

is holomorphic in the corresponding union $D_j \cup D_j^*$ and continuous when $x \in \partial D_{1,2} \cap \partial D_{1,2}^* \backslash \{-1, 1\}$ since $F_{1,2}^+(x) = A_{1,2}\upsilon_{1,2}(x) = \upsilon_3(x) + B_{1,2}\overline{\upsilon_{1,2}(x)} = F_{1,2}^-(x)$ by virtue of (1.1). This means that the function $F_1(z)$ is holomorphic in the unit circle $D^+ = \{z: |z| < 1\}$ and that $F_2(z)$ is holomorphic in $D^- = \{z: |z| < 1\}$.

By virtue of (2.1) and (2.2)

$$\begin{aligned} &\upsilon_j(z) = F_j(z) / A_j, && z \in D_j, \\ &\upsilon_3(z) = F_j(z) - \Delta_j \overline{F_j(\bar{z})}, && z \in D_j^*, \end{aligned} \quad j = 1,2 \tag{2.3}$$

It follows from (1.1), (1.2), (2.1) and (2.3) and the equality $\upsilon_3^+(t) = \upsilon_3^-(t)$, which holds on the semicircle $l^* = \{t: |t| = 1, \operatorname{Im} t < 0\}$, that the function $F(z) = \{F_1(z), z \in D^+; F_2(z), z \in D^-\}$ satisfies the conditions

$$F^+(t) = A_1 A_2^{-1} [AF^-(t) + Bt^{-2}\overline{F^-(t)}], \quad t \in l \tag{2.4}$$

$$F^+(t) - \Delta_1 \overline{F^+(\bar{t})} = F^-(t) - \Delta_2 \overline{F^-(\bar{t})}, \quad t \in l^*$$

$$F(\infty) = A_2 V_0 \tag{2.5}$$

Using relations (2.4) and the relations obtained from them by replacing t by $\bar{t}$ and the complex conjugation above the initial and transformed equalities, it is possible to show that the vector function $\Phi(z)$ with the components

$$\Phi_1(z) = F(z), \quad \Phi_2(z) = \overline{F(\bar{z})}, \quad \Phi_3(z) = F(1/z), \quad \Phi_4(z) = \overline{F(1/\bar{z})} \tag{2.6}$$

satisfies the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in l; \quad \Phi^+(t) = PG(t)P\Phi^-(t), \quad t \in l^* \tag{2.7}$$

The non-degenerate matrix $G(t)$ has the form

$$G(t) = \begin{Vmatrix} \delta_2 & 0 & 0 & \Delta t^{-2} \\ -\Delta & 1 & 0 & \Delta\Delta_1 t^{-2} \\ -\Delta\Delta_2 t^2 & 0 & 1 & \Delta \\ -\Delta t^2 & 0 & 0 & \delta_1 \end{Vmatrix}, \quad \det G(t) \equiv 1 \tag{2.8}$$

$$\delta_1 = 1 + \Delta\Delta_1, \quad \delta_2 = 1 - \Delta\Delta_2 \tag{2.9}$$

P is the permutation matrix

$$P = \begin{Vmatrix} J & O \\ O & J \end{Vmatrix}, \quad J = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad O = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix} \tag{2.10}$$

Henceforth, in carrying out the corresponding calculations, it is convenient to use the following pairwise-equivalent identities

$$\Delta\Delta_1\Delta_2 \equiv \Delta + \Delta_1 - \Delta_2 \tag{2.11}$$

$$\frac{\Delta_2 - \Delta}{\Delta_1} \equiv 1 - \Delta\Delta_2, \quad \frac{\Delta_1 + \Delta}{\Delta_2} \equiv 1 + \Delta\Delta_1 \tag{2.12}$$

which follow from (1.3) and (2.1). It follows from definitions (2.6) that it is necessary to solve problem (2.7) in the class of vector-functions which satisfy any two of the three conditions

$$\overline{\Phi(\bar{z})} \equiv P\Phi(z), \quad \Phi(1/z) \equiv Q\Phi(z), \quad \overline{\Phi(1/\bar{z})} \equiv R\Phi(z) \tag{2.13}$$

where Q and R, as well as P, are permutation matrices

$$Q = \begin{Vmatrix} O & I \\ I & O \end{Vmatrix}, \quad R = \begin{Vmatrix} O & J \\ J & O \end{Vmatrix}, \quad I = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \tag{2.14}$$

Integrable singularities are permitted at the points $z = \pm 1$ in the case of the components of $\Phi(z)$. Moreover, by virtue of (2.5) and (2.6)

$$\Phi_1(\infty) = \overline{\Phi_2(\infty)} = \Phi_3(0) = \overline{\Phi_4(0)} = A_2 V_0 \tag{2.15}$$

Using the substitutions

$$\Psi_{1,2}(z) = z\Phi_{1,2}(z), \quad \Psi_{3,4}(z) = \frac{1}{z}\Phi_{3,4}(z) \tag{2.16}$$

problem (2.7) can be reduced to a homogeneous Riemann problem in the vector function $\Psi(z)$ with the components (2.16)

$$\Psi^+(t) = T\Psi^-(t), \quad t \in l; \quad \Psi^+(t) = PTP\Psi^-(t), \quad t \in l^+ \tag{2.17}$$

where

$$T = G(1) \tag{2.18}$$

It follows from (2.16) that the solution of problem (2.17) has to be sought in the class of vector functions which, like the function $\Phi(z)$, satisfy conditions (2.13). Moreover, the first two components of the vector $\Psi(z)$ must have at least a simple zero at the origin of coordinates and a first-order pole at infinity, and inverse behaviour in the case of the third and fourth components of $\Psi(z)$. In view of (2.15) and (2.16)

$$-\operatorname{res}_\infty(\Psi_1(z)/z^2) = \operatorname{res}_0 \Psi_3(z) = A_2 V_0 \tag{2.19}$$

$$-\operatorname{res}_\infty(\Psi_2(z)/z^2) = \operatorname{res}_0 \Psi_4(z) = A_2 \overline{V_0}$$

On taking account of the equalities $T^{-1} = RTR$, $(PTP)^{-1} = QTQ$, which are easily verified using (2.1), (2.14) and (2.18), by means of the substitution

$$\Omega(z) = \begin{cases} \Psi(z), & z \in D^+ \\ PTP\Psi(z), & z \in D^- \end{cases} \tag{2.20}$$

we reduce problem (2.17) to the following

$$\Omega^+(t) = S\Omega^-(t), \quad t \in l \tag{2.21}$$

where, by virtue of (2.14) and (2.18), the matrix $S = TQTQ$ has the form

$$S = \begin{Vmatrix} \delta_2 & \Delta(\delta_1 + \delta_2 - 1) & \delta_2^2 - \delta_2 - \Delta^2 & \Delta \\ -\Delta & 1 + \delta_1^2 - \delta_1 - \Delta^2 & -\Delta(\delta_1 + \delta_2 - 1) & \delta_1 - 1 \\ \delta_2 - 1 & \Delta(\delta_1 + \delta_2 - 1) & 1 + \delta_2^2 - \delta_2 - \Delta^2 & \Delta \\ -\Delta & \delta_1^2 - \delta_1 - \Delta^2 & -\Delta(\delta_1 + \delta_2 - 1) & \delta_1 \end{Vmatrix} \tag{2.22}$$

Consequently, the vector function $\Omega(z)$ is holomorphic in the z plane with a cut along the semicircle l, in which it satisfies boundary condition (2.21) with a non-degenerate matrix (2.22) ($\det S = (\det T)^2 = 1$). By virtue of (2.13), (2.16) and (2.18)–(2.20), $\Omega(z)$ must also satisfy the conditions

$$\overline{\boldsymbol{\Omega}(\bar{z})} \equiv P\boldsymbol{\Omega}(z), \quad \boldsymbol{\Omega}(1/z) \equiv PTR\boldsymbol{\Omega}(z), \quad z \in D^+ \tag{2.23}$$

$$\mathrm{res}_\infty[z^{-2}\Omega(z)] = A_2((\Delta\overline{V_0} - V_0), \ (\Delta\Delta_2 - 1)\overline{V_0}, \ \Delta\overline{V_0}, \ \Delta\Delta_2\overline{V_0})$$

$$\mathrm{res}_0\, \Omega(z) = A_2(0, 0, V_0\overline{V_0},), \quad \mathrm{res}_0[z^{-1}\Omega_{1,2}(z)] = 0 \tag{2.24}$$

It follows from boundary condition (2.21) and the representation (2.22) that

$$\begin{aligned} \Omega_1^+(t) - \Omega_3^+(t) &= \Omega_1^-(t) - \Omega_3^-(t), \\ \Omega_2^+(t) - \Omega_4^+(t) &= \Omega_2^-(t) - \Omega_4^-(t), \end{aligned} \quad t \in l$$

This means that the arc l is not a line of discontinuity in the case of the functions $\Omega_1(z) - \Omega_3(z)$ and $\Omega_2(z)$–$\Omega_4(z)$. There are integrable and, consequently, removable singular points in the case of these functions at the points $z = \pm 1$. There are simple poles at zero and infinity in the case of these functions, that is, according to the generalized Liouville theorem

$$\Omega_1(z) - \Omega_3(z) = \alpha_1 z + \beta_1 + \gamma_1 / z$$

$$\Omega_2(z) - \Omega_4(z) = \alpha_2 z + \beta_2 + \gamma_2 / z$$

By virtue of relations (2.24)

$$\alpha_1 = -\mathrm{res}_\infty \frac{\Omega_1(z) - \Omega_3(z)}{z^2} = A_2 V_0, \quad \gamma_1 = \mathrm{res}_0(\Omega_1(z) - \Omega_3(z)) = -A_2 V_0$$

Using the second relation of (2.23) and (2.14) and (2.18), it is possible to obtain that $\Omega_1(z) - \Omega_3(z) = \Omega_3(1/z) - \Omega_1(1/z)$, whence it follows that $\beta_1 = 0$. In turn, we have

$$\overline{\Omega_1(\bar{z})} - \overline{\Omega_3(\bar{z})} = \Omega_2(z) - \Omega_4(z) = A_2\overline{V_0}(z - 1/z)$$

from the first condition of (2.23).

Thus

$$\begin{cases} \Omega_3(z) = \Omega_1(z) - A_2 V_0(z - 1/z) \\ \Omega_4(z) = \Omega_2(z) - A_2\overline{V_0}(z - 1/z) \end{cases} \tag{2.25}$$

On the basis of (2.25) and (2.22), the homogeneous four-dimensional problem (2.21) reduces to a two-dimensional inhomogeneous Riemann problem in the vector function $\mathbf{W}(z) = (W_1(z), W_2(z)) = (\Omega_1(z), \Omega_2(z))$

$$\mathbf{W}^+(t) = M\mathbf{W}^-(t) + \mathbf{m}(t), \quad t \in l \tag{2.26}$$

where the matrix M and the vector $\mathbf{m}(t)$ have the form

$$M = \begin{Vmatrix} \delta_2^2 - \Delta^2 & \Delta(\delta_1 + \delta_2) \\ -\Delta(\delta_1 + \delta_2) & \delta_1^2 - \Delta^2 \end{Vmatrix} \tag{2.27}$$

$$\mathbf{m}(t) = \Delta A_2(t - 1/t)((\Delta_2\delta_2 + \Delta)V_0 - \overline{V_0}, \ (\delta_1 + \delta_2 - 1)V_0 - \Delta_1\overline{V_0}) \tag{2.28}$$

On subtracting the first row of matrix (2.22) from the third row, and the second row from the fourth row and then adding the first column to the third column and the second column to the fourth column, we obtain $\det M = \det S = 1$. Consequently, the modulus of the eigenvalues of the matrix M is equal to unity which follows from the estimate

$$\begin{aligned} &-2 < -2\Delta^2 < \delta_1^2 + \delta_2^2 - 2\Delta^2 = \\ &= 2[1 - \Delta\Delta_2 + \Delta\Delta_1 - \Delta^2 + \Delta^2(\Delta_1^2 + \Delta_2^2)/2] < \\ &< 2(1 - \Delta\Delta_2 + \Delta\Delta_1) = 2[1 - \Delta^2(1 - \Delta_1\Delta_2)] < 2 \end{aligned}$$

which is established using (2.1), (2.11), (2.12) and (2.9). After some algebra, the required eigenvalues can be written in the following form

$$\mu_{1,2} = e^{\pm i2\pi\lambda} = \frac{(\delta_1+\delta_2)^2}{2} - 1 \pm i(\delta_1+\delta_2)\sqrt{1-\left(\frac{\delta_1+\delta_2}{2}\right)^2} \tag{2.29}$$

The eigenvector (h_1, h_2) of matrix (2.27), which corresponds to its first eigenvalue (2.29), found from the equation

$$(\delta_2^2 - \Delta^2 - \mu_1)h_1 + \Delta(\delta_1+\delta_2)h_2 = 0$$

can be taken in the form $(\bar{h}, h)$, where

$$h = e^{i\pi\gamma} = \frac{1}{2}\left(\sqrt{2+\Delta_1+\Delta_2} + i\,\mathrm{sign}\,\Delta\sqrt{2-\Delta_1-\Delta_2}\right) \tag{2.30}$$

The eigenvector which is the complex conjugate of the eigenvector which has been found corresponds to the second eigenvalue (2.29). It is clear that the matrix

$$H = \begin{Vmatrix} \bar{h} & h \\ h & \bar{h} \end{Vmatrix} \tag{2.31}$$

which reduces matrix (2.27) to normal Jordan form is non-degenerate since, by virtue of (2.30), $0 < |\gamma| < \pi/2$ in the case of the constraints which have been assumed in this section and, consequently, $\det H = -2i\sin(2\pi\gamma) \neq 0$.

Hence, after the linear substitution

$$\mathbf{W}(z) = zH\mathbf{V}(z) \tag{2.32}$$

problem (2.26) decomposes into the two one-dimensional problems

$$V_1^+(t) = \mu_1 V_1^-(t) + a_1(t^{-2}-1), \quad t \in l \tag{2.33}$$

$$V_2^+(t) = \mu_2 V_2^-(t) + a_2(t^{-2}-1), \quad t \in l \tag{2.34}$$

where

$$a_1 = -i\Delta A_2 \operatorname{cosec}(2\pi\gamma)e^{i\pi\lambda}\operatorname{Re}[V_0(\Delta_2\bar{h}-h)] \tag{2.35}$$

$$a_2 = i\Delta A_2 \operatorname{cosec}(2\pi\gamma)e^{-i\pi\lambda}\operatorname{Re}[V_0(\Delta_2 h-\bar{h})] \tag{2.36}$$

The right-hand sides of representations (2.35) and (2.36) are obtained on the basis of (2.28), (2.31), (2.32) and the relation

$$h(\delta_1+\delta_2-1) - \bar{h}(\Delta_2\delta_2+\Delta) = e^{i2\pi\lambda}(h-\Delta_1\bar{h})$$

which follows from equalities (2.9), (2.29) and (2.30).

It follows from (2.24) and (2.32) that the vector $\mathbf{V} = (V_1, V_2)$, with components which satisfy the conjugation conditions (2.33) and (2.34), must be bounded at zero and take the following value at infinity

$$\mathbf{V}(\infty) = A_2 H^{-1}(V_0 - \Delta\overline{V_0}, \delta_2\overline{V_0}) \tag{2.37}$$

The first condition of (2.23), when applied to the vector $\mathbf{V}(z)$ gives

$$\overline{H\mathbf{V}(\bar{z})} \equiv \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} H\mathbf{V}(z) \equiv \overline{H}\mathbf{V}(z), \quad z \in D^+$$

that is

$$\overline{\mathbf{V}(\bar{z})} \equiv \mathbf{V}(z), \quad z \in D^+ \tag{2.38}$$

Putting the second condition of (2.23) on the side for the present, we find the solution of problems (2.33) and (2.34) in the class of functions which satisfy conditions (2.37) and (2.38).

It is obvious that the single-valued branch of the analytic function

$$\chi(z)=\left(\frac{1+z}{1-z}\right)^{\lambda} \quad (\chi(0)=1) \tag{2.39}$$

which is fixed in the domain C/l, satisfies boundary condition (2.33) when $a_1 = 0$. Consequently, the inhomogeneous problem (2.33) reduces to a problem on a discontinuity in the function $V_1(z)\chi^{-1}(z)$

$$\frac{V_1^+(t)}{\chi^+(t)}=\frac{V_1^-(t)}{\chi^-(t)}+\frac{a_1}{\chi^+(t)}\frac{1-t^2}{t^2}, \quad t\in l \tag{2.40}$$

A particular solution of problem (2.40) gives an integral of the Cauchy type

$$I(z)=\frac{1}{2\pi i}\int_l \frac{1/t^2-1}{\chi^+(t)}\frac{dt}{t-z} \tag{2.41}$$

multiplied by the constant a_1. We now consider the auxiliary integral

$$I^*(z)=\frac{1}{2\pi i}\int_{l\cup l-} \frac{1/t^2-1}{\chi(t)}\frac{dt}{t-z}$$

On the one hand, the equality $I^*(z) = I(z)\,(1 - e^{i2\pi\lambda})$ holds since $\chi^-(t) = \exp(-i2\pi\lambda)\chi^+(t)$. On the other hand, by Cauchy's theorem on residues, the integral $I^*(z)$ is equal to the sum of the residues of the integrand $\varphi(t; z)$ at the points $t = 0$, $t = z$ and $t = \infty$, where, in the case of this integrand, there is a second order pole, a simple pole and a simple zero respectively. We find by standard methods

$$\operatorname{res}_0\varphi=\frac{2\lambda z-1}{z^2}, \quad \operatorname{res}_z\varphi=\frac{1/z^2-1}{\chi(z)}, \quad \operatorname{res}_\infty\varphi=e^{i\pi\lambda}$$

In calculating the last residue, use is made of the fact that, at infinity, the chosen branch of the functions (2.39) takes the value

$$\chi(\infty)=e^{-i\pi\lambda} \tag{2.42}$$

Hence

$$I(z)=\frac{1}{1-e^{i2\pi\lambda}}\left[e^{i\pi\lambda}-\frac{1}{\chi(z)}+\frac{1}{z^2}\left(\frac{1}{\chi(z)}-1+2\lambda z\right)\right] \tag{2.43}$$

In the neighbourhood of the zero for the chosen branch of the function $\chi^{-1}(z)$, the following Taylor series expansion holds

$$\chi^{-1}(z)=1-2\lambda z+2\lambda^2z^2+\ldots$$

This means that function (2.43) is holomorphic at zero and, by virtue of (2.42), it vanishes at infinity as must be the case.

Since $0 < \lambda < 1/2$, then, consequently, the product $V_1(z)\chi^{-1}(z)$ will have a removable singularity at the point $z = 1$ and, perhaps, just a simple pole at the point $z = -1$. This means that

$$V_1(z)=\chi(z)\left(a_1I(z)+\frac{c_{11}}{z+1}+c_{12}\right) \tag{2.44}$$

where c_{11}, c_{12} are arbitrary complex constants.

By virtue of (2.38), the function (2.44) must satisfy the condition

$$\overline{V_1(\bar{z})}\equiv V_1(z), \quad z\in D^+$$

Since the identity

$$\overline{\chi(\bar{z})} \equiv \begin{cases} \chi(z), & z \in D^+ \\ e^{i2\pi\lambda}\chi(z), & z \in D^- \end{cases} \tag{2.45}$$

holds for the chosen branch $\chi(z)$, then, by virtue of (2.43) and (2.44), in order to satisfy the last condition it is necessary that the following equalities should hold

$$1.\ \mathrm{Im}[a_1(1-e^{-i2\pi\lambda})]=0;\quad 2.\ \mathrm{Im}\,c_{11}=0;\quad 3.\ \mathrm{Im}\left[c_{12}+a_1\frac{e^{i\pi\lambda}}{1-e^{i2\pi\lambda}}\right]=0$$

Moreover, in view of (2.42)–(2.44) and (2.37), we must have

$$4.\ V_1(\infty)=c_{12}e^{-i\pi\lambda}=\frac{i}{2}A_2\,\mathrm{cosec}(2\pi\gamma)(\bar{h}V_0-(\Delta\bar{h}+\delta_2 h)\overline{V_0})$$

It is obvious that the first of these conditions is satisfied since, by virtue of (2.35)

$$\frac{a_1}{1-e^{i2\pi\lambda}}=\frac{2A_2}{4-(\Delta_1+\Delta_2)^2}\mathrm{Re}[(\Delta_2\bar{h}-h)V_0]$$

Using representations (2.29) and (2.30) and the relations

$$e^{i\pi\lambda}=\frac{\delta_1+\delta_2}{2}+i\,|\,\Delta\,|\sqrt{1-\left(\frac{\Delta_1+\Delta_2}{2}\right)^2} \tag{2.46}$$

$$e^{i2\pi\gamma}=\frac{\Delta_1+\Delta_2}{2}+i\,\mathrm{sign}\,\Delta\sqrt{1-\left(\frac{\Delta_1+\Delta_2}{2}\right)^2}$$

$$e^{i\pi\lambda}=\frac{h^2-\Delta_2}{h^2-\Delta_1}=-\frac{h^2-\Delta_2}{\bar{h}^2-\Delta_2}=-\frac{\bar{h}^2-\Delta_1}{h^2-\Delta_1}$$

which follow from them, it can be shown that the remaining necessary equalities 2–4 will hold if one puts

$$c_{12}=A_2e^{i\pi\lambda/2}\,\mathrm{cosec}(2\pi\gamma)\,\mathrm{Im}(e^{-i\pi\lambda/2}h\overline{V_0}),\quad c_{11}=c_1\in\mathbf{R}$$

After some simplifications using relations (2.9) and (2.46), the required solution of problem (2.33) can now be written in the form

$$V_1(z)=a(V_0)\left\{1-\frac{1}{z^2}+\chi(z)\left[1-\frac{2\lambda}{z}+\frac{1}{z^2}+\frac{c_1}{z+1}\right]\right\} \tag{2.47}$$

where

$$a(V_0)=(A_2/2)\sec(\pi\lambda/2)\,\mathrm{cosec}(2\pi\gamma)\,\mathrm{Im}(e^{-i\pi\lambda/2}h\overline{V_0}) \tag{2.48}$$

It can be shown in a similar manner that

$$V_2(z)=a(\overline{V_0})\left\{1-\frac{1}{z^2}+\frac{1}{\chi(z)}\left[1+\frac{2\lambda}{z}+\frac{1}{z^2}+\frac{c_2}{z-1}\right]\right\} \tag{2.49}$$

will be the corresponding solution of problem (2.34) with free term (2.36).

The arbitrary real constants c_1, c_2 in representations (2.47) and (2.49) have to be chosen in such a manner that the vector function

$$\mathbf{\Omega}(z)=(zH\mathbf{V}(z),\ zH\mathbf{V}(z)-A_2(z-1/z)(V_0,\ \overline{V_0}))$$

which is defined by relations (2.32) and (2.25), satisfies the second condition of (2.23). In order to do this, it is sufficient to require, for example, that the components of the vector $\Phi(z)$

$$\Phi_1(z)=\begin{cases}\bar{h}V_1(z)+hV_2(z), & z\in D^+\\ e^{i\pi\lambda}\bar{h}V_1(z)+e^{-i\pi\lambda}hV_2(z)+V_0A_2\Delta\Delta_2\left(1-\dfrac{1}{z^2}\right), & z\in D^-\end{cases}$$

$$\Phi_3(z)=\begin{cases}z^2[\bar{h}V_1(z)+hV_2(z)]+V_0A_2(1-z^2), & z\in D^+\\ z^2[e^{i\pi\lambda}\bar{h}V_1(z)+e^{-i\pi\lambda}hV_2(z)]+V_0A_2\delta_2(1-z^2), & z\in D^-\end{cases}$$

which have been found using the last equality and relations (2.20) and (2.16), should satisfy the condition $\Phi_1(1/\bar{z}) \equiv \Phi_3(z)$ in accordance with definition (2.6). On writing out this identity in explicit form for $z\in D^+$, using (2.47)–(2.49) and taking account of the last representations and the identity

$$\chi(1/z)\equiv\begin{cases}e^{-i\pi\lambda}\chi(z), & z\in D^+\\ e^{i\pi\lambda}\chi(z), & z\in D^-\end{cases} \tag{2.50}$$

we obtain $c_1 = c_2 = 0$, if, after all the reductions, one compares the coefficients of the homogeneous terms and, in particular, the coefficients of the terms $z\chi(z)$ and $z\chi^{-1}(z)$.

When account is taken of representations (2.3), (2.6) and (2.39) and the identity (2.45), it can be shown that, in all non-limiting and non-degenerate cases, the unique solution of problem (1.1), (1.2) is found using the formulae

$$\begin{aligned}
&\upsilon_1(z)=\frac{2k_1}{k_1+k_3}[e^{-i\pi\gamma}V_1(z)+e^{i\pi\gamma}V_2(z)]\\
&\upsilon_2(z)=\frac{2k_2}{k_2+k_3}[e^{i\pi(\lambda-\gamma)}V_1(z)+e^{-i\pi(\lambda-\gamma)}V_2(z)]+V_0\Delta\Delta_2\left(1-\frac{1}{z^2}\right)\\
&\upsilon_3(z)=(e^{-i\pi\gamma}-\Delta_1e^{i\pi\gamma})V_1(z)+(e^{i\pi\gamma}-\Delta_1e^{-i\pi\gamma})V_2(z)]\\
&V_1(z)=a(V_0)\left\{1-\frac{1}{z^2}+\chi(z)\left[1-\frac{2\lambda}{z}+\frac{1}{z^2}\right]\right\}\\
&V_2(z)=a(\bar{V}_0)\left\{1-\frac{1}{z^2}+\frac{1}{\chi(z)}\left[1+\frac{2\lambda}{z}+\frac{1}{z^2}\right]\right\}
\end{aligned} \tag{2.51}$$

and, in turn, if $V_0 = |V_0|e^{-i\alpha}$, then

$$a(V_0)=\frac{k_2+k_3}{4k_2}\frac{|V_0|\sin[\pi(\gamma-\lambda/2+\alpha)]}{\cos(\pi\lambda/2)\sin(2\pi\gamma)}$$

$$\lambda=\frac{1}{\pi}\arccos\left(1+\Delta\frac{\Delta_1-\Delta_2}{2}\right),\quad \gamma=\frac{\operatorname{sign}\Delta}{2\pi}\arccos\frac{\Delta_1+\Delta_2}{2}$$

$$\Delta=\frac{k_2-k_1}{k_2+k_1},\quad \Delta_1=\frac{k_1-k_3}{k_1+k_3},\quad \Delta_2=\frac{k_2-k_3}{k_2+k_3}$$

Remark 1. We recall that the functions $\upsilon_j(z)$ which have been found are complex conjugate with the values of the true seepage rates $\upsilon_j(z)$. Working with the above formulae, it is necessary to take account of the fact that, first, the inequalities

$$0<\arg\chi<\pi\lambda/2,\quad -3\pi\lambda/2<\arg\chi<-\pi\lambda,\quad -\pi\lambda<\arg\chi<0,$$

hold when $z\in D_1$, $z\in D_2$ and $z\in D_3$ respectively for the chosen branch of the multivalued function (2.39) in accordance with the method of setting it and, second, the following Taylor series expansions hold

$$\chi(z)=1+2\lambda z+2\lambda^2z^2+\frac{2\lambda}{3}(1+2\lambda^2)z^3+\frac{2\lambda^2}{3}(2+\lambda^2)z^4+\ldots,\quad |z|<1$$

$$\chi(z)=e^{-i\pi\lambda}(1+2\lambda z^{-1}+2\lambda^2z^{-2}+\frac{2\lambda}{3}(1+2\lambda^2)z^{-3}+\ldots),\quad |z|>1$$

The expansion in the corresponding Taylor series for the function $\chi^{-1}(z)$ is obtained from this by simply replacing λ by $-\lambda$.

Remark 2. The complex potential $w(z) = w_j(z), z\in D_j, (j = 1, 2, 3)$

$$
\begin{aligned}
w_1(z) &= \frac{2k_1}{k_1+k_3}[e^{-i\pi\gamma}W_1(z)+e^{i\pi\gamma}W_2(z)] \\
w_2(z) &= \frac{2k_2}{k_2+k_3}[e^{i\pi(\lambda-\gamma)}W_1(z)+e^{-i\pi(\lambda-\gamma)}W_2(z)]+V_0\Delta\Delta_2\left(z+\frac{1}{z}\right) \\
w_3(z) &= (e^{-i\pi\gamma}-\Delta_1 e^{i\pi\gamma})W_1(z)+(e^{i\pi\gamma}-\Delta_1 e^{-i\pi\gamma})W_2(z)] \\
W_1(z) &= a(V_0)\left\{z+\frac{1}{z}+\chi(z)\left(z-\frac{1}{z}\right)\right\} \\
W_2(z) &= a(\overline{V}_0)\left\{z+\frac{1}{z}+\frac{1}{\chi(z)}\left(z-\frac{1}{z}\right)\right\}
\end{aligned}
\tag{2.52}
$$

is easily re-established by the integration of formulae (2.51).

The cases of degeneration of the three-component medium into a two-component medium ($k_j = k_m$) as well as the limiting cases when at least one of the components of the medium D_j is an ideally conducting medium ($k_j = \infty$) or an impermeable medium ($k_j = 0$), which are of interest from the point of view of seepage applications, are considered below.

3. THE SOLUTION OF PROBLEM (1.1), (1.2) IN DEGENERATE CASES

In almost all of the situations which are investigated in this and the following sections, with the exception of those which are specially mentioned, the required solutions and the corresponding complex potential can be obtained using formulae (2.51) and (2.52) by taking the limit in relations (1.3), (2.1), (2.9), (2.30), (2.39), (2.46) and (2.48). Hence, as a rule, in each actual case only the final result of taking the limit in the above mentioned formulae will be indicated.

3.1. *A single semicircular inclusion.* It is obvious that the medium shown in Fig. 2(a) is obtained from the three-component medium considered above when $k_2 = k_3$. It extends the well-known scheme with a circular inclusion [3] and, in view of the existence of a second geometrical parameter (the angle of inclination of the base of the inclusion to the external flow vector) in addition to the radius, it can serve to construct stochastic models of the classical Maxwellian effective medium type [16].

If $k_2 = k_3$, then $A_2 = 1$, $\Delta_2 = 0$, $\Delta_1 = -\Delta$, $\delta_1 = 1 - \Delta^2$, $\delta_2 = 1$ and

$$
\lambda = 4\gamma - \operatorname{sign}\Delta = \frac{1}{\pi}\arccos\left(1-\frac{\Delta^2}{2}\right),\quad a(V_0) = \frac{\operatorname{Re}(hV_0)}{(4-\Delta^2)}
\tag{3.1}
$$

Here, it is obvious that $\bar{h} - \Delta_1 h = \bar{h}\exp(i\pi\lambda)$ and this means that $\upsilon_2(z) \equiv u_3(z)$, which must be so in the given case.

3.2. *The problem of a semicircular groove.* On putting $k_1 = k_3$, we arrive at the two-component porous medium shown in Fig. 2(b). Such a medium can be considered when simulating the process of channel-shaped undermining [17] when the groove simulates an incipient erosion finger.

Here, $A_1 = 1$, $A_2 = A$, $\Delta_1 = 0$, $\Delta_2 = \Delta$, $\delta_1 = 1$, $\delta_2 = 1 - \Delta^2$ and

$$
\lambda = \operatorname{sign}\Delta - 4\gamma = \frac{1}{\pi}\arccos\left(1-\frac{\Delta^2}{2}\right),\quad a(V_0) = -A\frac{\operatorname{Re}(h^3\overline{V}_0)}{4-\Delta^2}
\tag{3.2}
$$

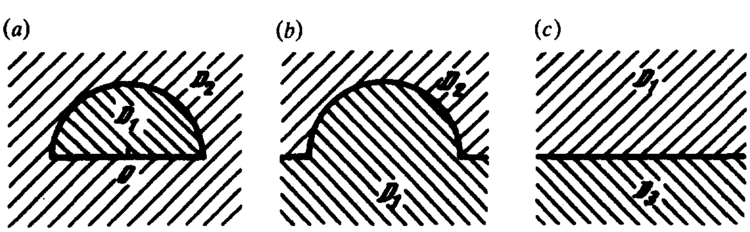

Fig. 2.

(The identity $\upsilon_1(z) \equiv \upsilon_3(z)$ is obvious in this case.)

3.3. *The problem of two half-planes.* In the simplest case (Fig. 2c) when $k_1 = k_2$, it is easy to obtain $A_1 = A_2$, $\Delta = 0$, $\Delta_1 = \Delta_2$, $\delta_1 = \delta_2 = 1$, $\lambda = 0$ and, as it also must be [3]

$$\upsilon_1(z) \equiv \upsilon_2(z) \equiv V_0, \quad \upsilon_3(z) \equiv k_3 V_{0x}/k_1 - iV_{0y}$$

4. THE SOLUTION OF PROBLEM (1.1), (1.2) IN LIMITING CASES

It should be noted that, in the theory of seepage, the coefficient of proportionality k_j in Darcy's law has an upper limit. Hence, in the case of a value of k_i which is above the limiting value, it is customary not to consider the seepage in the corresponding component D_j, but to take the boundary of this component as an equipotential. However, in other sections of the theory of heterogeneous media, large values of the conductivity are permissible and the field in an ideally conducting component may be of interest.

4.1. *The cases when* $k_1 = 0, \infty$.

(a) $k_1 = \infty$, $k_2 \neq k_3$, $k_{2,3} \neq 0, \infty$:

$A_1 = {}^1/_2$, $\Delta = -\Delta_1 = -1$, $\delta_1 = 0$, $\delta_2 = 1 + \Delta_2$

$$\lambda = -2\gamma = \frac{1}{\pi}\arccos\frac{1+\Delta_2}{2}, \quad a(V_0) = \frac{2A_2\,\mathrm{Im}(e^{i\pi\lambda}V_0)}{(3+\Delta_2)\sqrt{1-\Delta_2}} \tag{4.1}$$

(a′) $k_1 = \infty$, $k_2 = k_3 \neq 0, \infty$: $A_2 = 1$, $\Delta_2 = 0$ and it follows from (4.1) that $\lambda = -2\gamma = 1/3$, $a(V_0) = 2\mathrm{Im}(e^{i\pi/3}V_0)/3$ exactly as in (3.1)

(b) $k_1 = 0$, $k_2 \neq k_3$, ($k_{2,3} \neq 0, \infty$):

$A_1 = \infty$, $\Delta = -\Delta_1 = -1$, $\delta_1 = 0$, $\delta_2 = 1 + \Delta_2$

$$\lambda = 1 - 2\gamma = \frac{1}{\pi}\arccos\frac{1-\Delta_2}{2}, \quad a(V_0) = \frac{2A_2\,\mathrm{Re}(e^{i\pi\lambda}V_0)}{(3-\Delta_2)\sqrt{1+\Delta_2}} \tag{4.2}$$

(b′) $k_1 = 0$, $k_2 = k_3 \neq 0, \infty$: $\upsilon_1(z) \equiv 0$, $A_2 = 1$, $\Delta_2 = 0$ and, by virtue of (4.2), $\lambda = \gamma = 1/3$, $a(V_0) = 2\mathrm{Re}(e^{i\pi/3}V_0)/3$.

4.2. *The cases when* $k_2 = 0, \infty$.

(a) $k_2 = \infty$, $k_1 \neq k_3$, $k_{1,3} \neq 0, \infty$:

$A = A_1 = 1/2$, $\Delta = \Delta_2 = 1$, $\delta_1 = 1 + \Delta_1$, $\delta_2 = 0$

$$\lambda = 2\gamma = \frac{1}{\pi}\arccos\frac{1+\Delta_1}{2}, \quad a(V_0) = \frac{\mathrm{Im}\,\overline{V}_0}{(3+\Delta_1)\sqrt{1-\Delta_1}} \tag{4.3}$$

(a′) $k_2 = \infty$, $k_1 = k_3 \neq 0, \infty$: from (3.2) and from (4.3) it equally follows that $\lambda = 2\gamma = 1/3$, $a(V_0) = \mathrm{Im}\overline{V}_0/3$.

(b) In the case when $k_2 = 0$, $k_1 \neq k_3$, $k_{1,3} \neq 0, \infty$, the solution of the corresponding boundary-value problem cannot be obtained by directly taking the limit, which is completely understandable in view of the fact that the physical meaning in this situation must be $\upsilon_2 \equiv 0$ and this contradicts condition (1.2). Such a difficulty would not arise if one were to specify that $\upsilon_3(\infty) = V_0^* = V_{0x}^* - iV_{0y}^*$ instead of (1.2). In the general case, it follows from (1.1) and (1.2) that $V_0 = k_2V_{0x}^*/k_3 - iV_{0y}^*$. In particular, if $V_{0y} = V_{0y}^* = 0$, then $V_0 = V_{0x} = k_2V_{0x}^*/k_3$. On now replacing V_0 by $k_2V_{0x}^*/k_3$ in all the formulae and taking the limit when $k_2 \to 0$, it can be found that $\Delta = \Delta_2 = -1$, $A = A_2 = \infty$, $\delta_1 = 1 - \Delta_1$, $\delta_2 = 0$

$$\lambda = 1 + 2\gamma = \frac{1}{\pi}\arccos\frac{1-\Delta_1}{2}, \quad a(V_0) = \frac{V_{0x}^*}{(3-\Delta_1)\sqrt{1+\Delta_1}} \tag{4.4}$$

(b′) In turn, formulae (4.4) when $k_2 = 0$ and $k_1 = k_3 \neq 0, \infty$ give: $\Delta_1 = 0$ and $\lambda = -\gamma = 1/3$, $a(V_0) = V_{0x}^*/3$.

4.3. *The cases when* $k_3 = 0, \infty$.

(a) In the limit when $k_3 \to \infty$, $k_1 \neq k_2$, $k_{1,2} \neq 0, \infty$, $\mathrm{Re}\upsilon_{1,2}(x) = 0$ follows from the condition that $k_3\mathrm{Re}\upsilon_{1,2}(x) = k_{1,2}\mathrm{Re}\upsilon_3(x) = x \in \partial D_{1,2} \geqslant \partial D_3$

$$v_2(\infty) = V_0 = -iV_{0y} \tag{4.5}$$

It is obvious that the functions

$$F_{1,2}(z) = \begin{cases} v_{1,2}(z), & z \in D_{1,2} \\ -\overline{v_{1,2}(\bar{z})}, & z \in D^*_{1,2} \end{cases}$$

are holomorphic in the domains D^+, D^-, respectively, and that the function

$$F(z) = \begin{cases} \dfrac{k_2}{k_1} AF_1(z), & z \in D^+ \\ F_2(z) + \dfrac{k_2}{k_1} B\dfrac{1}{z^2}\overline{F_1(1/\bar{z})}, & z \in D^- \end{cases}$$

is holomorphic in the extended plane z in view of the boundary equality

$$v_2(t) = \frac{k_2}{k_1} Av_1(t) - \frac{k_2}{k_1} Bt^{-2}\overline{v_1(t)}, \quad |t| = 1$$

which follows from the first boundary condition of (1.1). According to Liouville's theorem, $F(z) \equiv$ const. and, by virtue of (4.5)

$$v_1(z) = -iV_{0y}(1-\Delta), \quad v_2(z) = -iV_{0y}(1+\Delta z^{-2}) \tag{4.6}$$

It is possible to arrive at the same result if, starting out from the corresponding formulae in Section 2, it is shown that

$$\lim_{k_3\to\infty} A_1^{-1}a(V_0) = \frac{V_{0y}(1-\Delta)}{4}\operatorname{sign}\Delta, \quad \lim_{k_3\to\infty} A_2^{-1}a(V_0) = \frac{V_{0y}(1+\Delta)}{4}\operatorname{sign}\Delta$$

and account is taken of the fact that $\Delta_1 = \Delta_2 = -1$, $\lambda = 0$, $2\gamma = \operatorname{sign}\Delta$.

The field in the half-plane D_3 can be found by solving the Schwartz boundary-value problem

$$\mathrm{Im}v_3(x) = \mathrm{Im}v_j(x), \quad x \in \partial D_j \cap \partial D_3, \quad j = 1, 2$$

the free term of which is determined by the equalities (4.6). Omitting the calculations, we merely present the final result

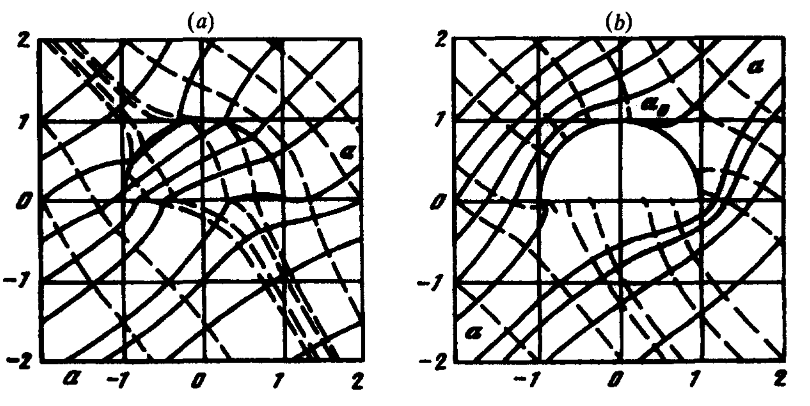

Fig. 3.

$$\upsilon_3(z) = -iV_{0y}\left\{1 - \Delta + \frac{\Delta}{i\pi}\left[\left(1 + \frac{1}{z^2}\right)\ln\frac{1-z}{1+z} + \frac{2}{z}\right]\right\} \tag{4.7}$$

(under the logarithm, the vanishing branch, fixed in D_3, is understood). We note that the corresponding complex potential in the case being considered will have the form

$$w_1(z) = -iV_{0y}(1-\Delta)z, \quad w_2(z) = -iV_{0y}(z - \Delta z^{-1})$$

$$w_3(z) = -iV_{0y}\left\{(1-\Delta)z + \frac{\Delta}{i\pi}\left(z - \frac{1}{z}\right)\ln\frac{1-z}{1+z}\right\}$$

(a′) If $k_3 = \infty$ and $k_1 = k_2 \neq 0, \infty$ then $\Delta = 0$, and it follows from (4.6) and (4.7) that $\upsilon_1 \equiv \upsilon_2 \equiv \upsilon_3 \equiv iV_{0y}$.

(b) When $k_3 = 0, k_1 \neq k_2, k_{1,2} \neq 0, \infty$, it is obvious that the conditions $\operatorname{Im}\upsilon_j(x) = 0$ ($j = 1, 2, 3$) must be satisfied. This means that $\upsilon_3(z) \equiv 0$ and $\upsilon_{2\infty} = V_{0x}$. As in case (a) we find

$$\upsilon_1(z) = V_{0x}(1-\Delta), \quad \upsilon_2(z) = V_{0x}(1 - \Delta z^{-2}) \tag{4.8}$$

(b′) In the trivial case when $k_3 = 0$ and $k_1 = k_2$, it follows from (4.8) that $\upsilon_1 \equiv \upsilon_2 \equiv V_{0x}$.

The exact solutions which have been obtained enable us to reproduce the structure of the seepage field in the domain where it is substantially two-dimensional. The stream lines and equipotential lines (the solid curves and the dotted curves, respectively) are shown in Fig. 3(a, b) in the case when the external field is orientated at an angle of 45° to the abscissa. In the case of the draining lens ($k_1 = 10$, $k_2 = 0.6, k_3 = 1$, Fig. 3a), the behaviour of the stream lie a–a, which makes a "loop" close to the angle of the lens, is interesting. In the case of a weakly permeable barrier ($k_1 = 0.01, k_2 = k_3 = 1$, Fig. 3b), the behaviour of the stream line a–a, which "separates" in the rear part of the lens (the point a_0), should be noted.

We emphasize that standard methods of the finite-element type or the method of finite differences, which are based on numerical differentiation of the mesh point values of the support, are bad at describing such non-trivial effects [18]. The technique of tracking labelled particles, which has been used above, enables one to start from any point in a medium, and the errors which arise in the numerical integration of the system of two differential equations with specified, analytic right-hand sides can also be estimated analytically. Although the domains of non-trivial distribution of the flow parameters are small and, globally, the behaviour of the field satisfies that which is intuitively expected, the real refraction picture also enables one to estimate the validity of the packages of approximate models, for example, the condition that the substrate in the problem of the dissipation of heat from a finned surface should be isothermal [18, 19]. Such distributions as isotachs [3], isochrones [20] and advective transfer concentration curves [21] (which are easily constructed using the solution which has been obtained) can be employed for geomechanical and hydrogeological analysis. For example, the velocity distribution along a semicircle ("a centre of erosion") is important when describing the formation of the dendritic structures of "undermining fingers" [17]. In models of convective diffusion [22], it is important to evaluate the validity of the asymptotic forms and of the approximate models of the Numerov–Patrashev type by estimating the dimensions of the zones of extremely non-constant velocity. The integral characteristics are also of interest, namely, the total dissipation, the effective conductance, the total mass flow from the lens etc. However, these questions are outside the scope of this paper.

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